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Novel relations and new properties of confluent Heun's functions and their derivatives of arbitrary order

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Abstract
The present paper reveals important properties of the confluent Heun's functions. We derive a set of novel relations for confluent Heun's functions and their derivatives of arbitrary order. Specific new subclasses of confluent Heun's functions are introduced and studied. A new alternative derivation of confluent Heun's polynomials is presented.

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1. Introduction

The solutions of the confluent Heun's differential equation [1–6], written in the simplest uniform shape [7]

$$H'' + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right) H' + \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right) H = 0, \tag{1.1}$$

are of continuous and significant interest for many applications in different areas of natural sciences and especially in physics.

Equation (1.1) has three singular points: two regular ones—$z = 0$ and $z = 1$—and one irregular one—$z = \infty$. The standard confluent Heun's function HeunC$(\alpha, \beta, \gamma, \delta, \eta, z)$ is a unique particular solution, which is regular around the regular singular point $z = 0$. It is defined via the convergent in the disc $|z| < 1$ Taylor series expansion

$$\mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z) = \sum_{n=0}^{\infty} v_n(\alpha, \beta, \gamma, \delta, \eta) z^n, \tag{1.2}$$

assuming the normalization HeunC$(\alpha, \beta, \gamma, \delta, \eta, 0) = 1$. The parameters $\alpha, \beta, \gamma, \delta, \eta$, introduced in [3, 4] and used in the widespread computer package Maple, are related with μ

and ν according to the equations $\mu = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \eta$ and $\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha\gamma + \beta\gamma) + \delta + \eta$.

The coefficients $v_n(\alpha, \beta, \gamma, \delta, \eta)$ are determined by the three-term recurrence relation

$$A_n v_n = B_n v_{n-1} + C_n v_{n-2}, \tag{1.3}$$

with the initial condition $v_{-1} = 0,\ \ v_0 = 1$. Here,

$$A_n = 1 + \frac{\beta}{n} \to 1, \quad \text{when } n \to \infty,$$
$$B_n = 1 + \frac{-\alpha + \beta + \gamma - 1}{n} + \frac{\eta - (-\alpha + \beta + \gamma)/2 - \alpha\beta/2 + \beta\gamma/2}{n^2} \to 1, \quad \text{when} \quad n \to \infty,$$
$$C_n = \frac{\alpha}{n^2}\left(\frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + n - 1\right) \to 0, \quad \text{when } n \to \infty. \tag{1.4}$$

According to [1–6] the function $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$ reduces to a polynomial of degree $N \geqslant 0$ with respect to the variable z if and only if the following two conditions are satisfied:

$$\frac{\delta}{\alpha} + \frac{\beta + \gamma}{2} + N + 1 = 0, \tag{1.5a}$$
$$\Delta_{N+1}(\mu) = 0. \tag{1.5b}$$

Further on we call the first condition (1.5*a*)—a 'δ_N-condition'—and the second one (1.5*b*)—a 'Δ_{N+1}-condition'. One can find an explicit form of the left-hand side $\Delta_{N+1}(\mu)$ of condition (1.5*b*), convenient for practical calculations, in the appendix.

Indeed, the δ_N-condition is equivalent to the equation $C_{N+2} = 0$, and the Δ_{N+1}-condition turns to be equivalent to the requirement $v_{N+1}(\alpha, \beta, \gamma, \delta, \eta) = 0$. Then as a result of equation (1.3) and additional conditions (1.5) we obtain $v_{N+2}(\alpha, \beta, \gamma, \delta, \eta) = 0$. Since two consecutive terms in the tree-term recurrence relation (1.3) are zero, all next terms are zero, too. Hence, under simultaneous fulfilment of the two additional conditions (1.5), the confluent Heun function $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$ (1.2) reduces to a polynomial of degree N.

In this paper we derive a set of novel relations and differential equations for confluent Heun's functions (1.2) and their derivatives $\frac{d^n}{dz^n}\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$ of arbitrary order n, see section 2. In section 3 we introduce a new subclass of confluent Heun's functions $\text{HeunC}_N(\alpha, \beta, \gamma, \eta, z)$, which obey only the δ_N-condition (1.5*a*), as well as their associate confluent Heun's functions $\text{HeunC}_N^{\maltese}(\alpha, \beta, \gamma, \eta, z)$. In section 4 we utilize the newly found relations to present an alternative derivation of the confluent Heun's polynomials without use of the recurrence relation (1.3). Thus, our consideration reveals important new properties of the confluent Heun functions.

In the concluding section 5 we briefly discuss the relation of our results for confluent Heun's functions with similar results for general Heun's functions.

The confluent Heun's functions $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$, the specific classes of δ_N-confluent Heun's functions $\text{HeunC}_N(\alpha, \beta, \gamma, \eta, z)$ and their associated functions $\text{HeunC}_N^{\maltese}(\alpha, \beta, \gamma, \eta, z)$, as well as confluent Heun's polynomials $\text{PHeunC}_{N,k}(\alpha, \beta, \gamma, z)$, play a very important role in some applications, especially in gravitational physics [7–16]. The present study of their properties was inspired by our desire to reach a true mathematical and physical understanding of the important Teukolsky–Starobinsky identities, derived and used in the early articles [17–19]. We present here the formal mathematical results, which can be applied, too, in other scientific domains, both for analytical and for numerical calculations.

2. Novel relations for the confluent Heun's functions and their derivatives

Let us define the differential expression

$$\hat{D}_{\alpha,\beta,\gamma,\delta,\eta} = z(z-1)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(\alpha + \frac{\beta+1}{z} + \frac{\gamma+1}{z-1}\right)\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\mu}{z} + \frac{\nu}{z-1}\right). \tag{2.1}$$

One can use it to write down the confluent Heun equation (1.1) in the following compact form:

$$\hat{D}_{\alpha,\beta,\gamma,\delta,\eta} H = 0. \tag{2.2}$$

It can be shown that certain restrictions of the differential expression $\hat{D}_{\alpha,\beta,\gamma,\delta,\eta}$ on proper functional spaces yield self-adjoint differential operators [5]. In the present paper we will skip the detail, needed for justification of the operators' domains and proper scalar products in the corresponding linear spaces of functions. Here, we restrict our consideration only to formal manipulations with differential expressions like (2.1).

The confluent Heun's operator $\hat{D}_{\alpha,\beta,\gamma,\delta,\eta}$ (2.1) owns a remarkable property. Its eigenfunctions $H_\lambda(z)$ for eigenvalues $\lambda \neq 0$, i.e. the solutions of the ordinary differential equation

$$\hat{D}_{\alpha,\beta,\gamma,\delta,\eta} H_\lambda(z) = \lambda H_\lambda(z), \tag{2.3}$$

are at the same time solutions of confluent Heun's equation (2.2) with the same parameters $\alpha, \beta, \gamma, \delta$ and a different parameter $\eta' = \eta - \lambda$. For them the following confluent Heun's equation takes place:

$$\hat{D}_{\alpha,\beta,\gamma,\delta,\eta-\lambda} H_\lambda(z) = 0, \qquad \forall\, \lambda \in \mathbb{C}. \tag{2.4}$$

Indeed, equation (2.3) obviously can be rewritten in the form (2.2) with $\mu' = \mu + \lambda$ and $\nu' = \nu - \lambda$. Then, using the relations $\delta = \nu + \mu - \alpha\left(\frac{\beta+\gamma}{2} - 1\right)$ and $\eta = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \mu$ one obtains the above result.

Commuting $\hat{D}_{\alpha,\beta,\gamma,\delta,\eta}$ with the differential expression $\frac{\mathrm{d}^n}{\mathrm{d}z^n}$ we derive the basic novel relation

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\hat{D}_{\alpha,\beta,\gamma,\delta,\eta} = \hat{D}_{\alpha(n),\beta(n),\gamma(n),\delta(n),\eta(n)}\frac{\mathrm{d}^n}{\mathrm{d}z^n} + n\alpha\left(\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + n\right)\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}, \qquad n = 0, 1, 2, \ldots. \tag{2.5}$$

The transformation $\{\alpha, \beta, \gamma, \delta, \eta\} \to \{\alpha(n), \beta(n), \gamma(n), \delta(n), \eta(n)\}$ defined by the relations

$$\alpha(n) = \alpha, \qquad \beta(n) = \beta + n, \qquad \gamma(n) = \gamma + n, \qquad \frac{\delta(n)}{\alpha(n)} = \frac{\delta}{\alpha} + n,$$
$$\eta(n) = \eta + \frac{n}{2}(n - \alpha + \beta + \gamma) \tag{2.6}$$

yields a specific augmentation of the indices $\alpha, \beta, \gamma, \delta$ and η.

To simplify our notations we will use further on the following ones:

$$\hat{D}_{\alpha,\beta,\gamma,\delta,\eta} \equiv \hat{D}_0, \qquad \hat{D}_{\alpha(1),\beta(1),\gamma(1),\delta(1),\eta(1)} \equiv \hat{D}_1, \ldots, \hat{D}_{\alpha(n),\beta(n),\gamma(n),\delta(n),\eta(n)} \equiv \hat{D}_n, \ldots. \tag{2.7}$$

Applying equation (2.5) to the arbitrary solution $H(z)$ of confluent Heun's equation (2.2) we obtain the following four-term recurrence relation for the derivatives $\frac{\mathrm{d}^n}{\mathrm{d}z^n}H(z) = H^{(n)}(z)$:

$$\hat{D}_n H^{(n)}(z) = -n\alpha\left(\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + n\right)H^{(n-1)}(z), \qquad n = 0, 1, 2, \ldots. \tag{2.8}$$

For $n \geqslant 2$ it can be considered, too, as an ordinary differential equation of third order for the functions $H^{(n-1)}(z)$.

Then applying several times equation (2.8) we obtain the following sequence of relations for the arbitrary solution $H(z)$ of confluent Heun's equation (2.2):

$$\hat{D}_1\hat{D}_2\cdots\hat{D}_n H^{(n)}(z) = (-\alpha)^n n!\left(\frac{\delta}{\alpha}+\frac{\beta+\gamma}{2}+1\right)_n H(z), \qquad n = 1, 2, \ldots. \tag{2.9}$$

Here, $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\cdots(x+n-1)$ stands for the Pochhammer symbol [2].

Finally, applying $\hat{D}_0$ to both sides of equation (2.9) we end with the ordinary differential equations of order $2(n+1)$ for derivatives $H^{(n)}(z)$ of the solution $H(z)$ of confluent Heun's equation (2.2):

$$\hat{D}_0\hat{D}_1\cdots\hat{D}_n H^{(n)}(z) = 0, \qquad n = 1, 2, \ldots. \tag{2.10}$$

Relations (2.9) obviously show that any solution $H(z)$ of confluent Heun's equation (2.2) is an eigenfunction of the operator $\hat{D}_1\hat{D}_2\cdots\hat{D}_n\frac{\mathrm{d}^n}{\mathrm{d}z^n}$ and $(-\alpha)^n n!\left(\frac{\delta}{\alpha}+\frac{\beta+\gamma}{2}+1\right)_n$ is the corresponding eigenvalue. Relations (2.10) show that the solutions $H(z)$ of the confluent Heun's equation belong simultaneously to the null spaces of the infinite sequence of linear operators $\hat{D}_0\hat{D}_1\cdots\hat{D}_n\frac{\mathrm{d}^n}{\mathrm{d}z^n}$: $n = 0, 1, 2\ldots$.

3. New subclass of confluent Heun's functions

When the δ_N-condition (1.5*a*) is fulfilled, one obtains $\delta = -\alpha\left(\frac{\beta+\gamma}{2}+N+1\right) = \delta_N$ for some fixed nonnegative integer $N \geqslant 0$. Then for any function $H(z) \in \mathcal{C}^{N+3}$ equation (2.5) reduces to

$$\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}}(\hat{D}_0 H) = \hat{D}_{N+1}\left(\frac{\mathrm{d}^{N+1}H}{\mathrm{d}z^{N+1}}\right). \tag{3.1}$$

This relation shows that in the case under consideration the operator $\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}}$ defines a specific generalized Darboux transformation [20] for the confluent Heun's operator $\hat{D}_0$. It is remarkable that on the right-hand side of equation (3.1) we have just another confluent Heun's operator—$\hat{D}_{N+1}$. As a result, if the function $H_\lambda(z)$ with $\delta = \delta_N$ is an eigenfunction of the confluent Heun operator $\hat{D}_0$ for some eigenvalue λ, then relation (3.1) shows that the function $\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}} H_\lambda(z)$ is an eigenfunction to the corresponding operator $\hat{D}_{N+1}$ for the same eigenvalue λ.

Note that the condition $\delta = \delta_N$ is invariant under arbitrary changes of the eigenvalue λ. The operator $\hat{D}_0$ with $\delta = \delta_N$ preserves the linear envelope of its eigenfunctions with different λ, with the same $\delta = \delta_N$ and fixed nonnegative integer N. The discussion of equation (2.3) at the beginning of section 2 shows that studying this linear space of functions, further on we can restrict our consideration to the case $\lambda = 0$. All formulae for $\lambda \neq 0$ can be obtained from the ones for $\lambda = 0$ replacing η with $\eta - \lambda$.

Suppose the function $H(z)$ is in addition a solution of equation (2.2). Then relation (3.1) yields the following novel confluent Heun's equation for the derivative $\frac{\mathrm{d}^{N+1}H}{\mathrm{d}z^{N+1}}$:

$$\hat{D}_{N+1}\left(\frac{\mathrm{d}^{N+1}H}{\mathrm{d}z^{N+1}}\right) = 0. \tag{3.2}$$

Consider the two unique solutions of Heun's differential equations (2.2) and (3.2), which are simultaneously regular at the point $z = 0$ and equated to unity at this point. Then these solutions are precisely the two confluent Heun's functions HeunC$(\alpha, \beta, \gamma, \delta_N, \eta, z)$ and HeunC$(\alpha(N+1), \beta(N+1), \gamma(N+1), \delta_N(N+1), \eta(N+1), z)$, the last being constructed according to equations (2.6). For them the uniqueness theorem and the uniform normalization at the point $z = 0$ yield the novel relation[1]

$$\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}}\mathrm{HeunC}(\alpha,\beta,\gamma,\delta_N,\eta,z) = (N+1)!\,v_{N+1}(\alpha,\beta,\gamma,\delta_N,\eta) \times \mathrm{HeunC}(\alpha(N+1),\beta(N+1),\gamma(N+1),\delta_N(N+1),\eta(N+1),z). \tag{3.3}$$

Here, we have used the $(N+1)$st coefficient in the Taylor series expansion (1.2).

Definition 1. *We call δ_N-confluent-Heun's functions the functions $HeunC(\alpha,\beta,\gamma,\delta_N,\eta,z)$ which obey the δ_N-condition for some specific nonnegative integer $N \geqslant 0$ and denote them as*

$$\mathrm{HeunC}_N(\alpha,\beta,\gamma,\eta,z) = \mathrm{HeunC}(\alpha,\beta,\gamma,-\alpha\,(N+1+(\beta+\gamma)/2)\,,\eta,z). \tag{3.4}$$

Definition 2. *We call associate δ_N-confluent-Heun's functions the functions*

$$\mathrm{HeunC}_N^{\maltese}(\alpha,\beta,\gamma,\eta,z) \tag{3.5}$$

$$= \mathrm{HeunC}(\alpha,\beta+N+1,\gamma+N+1,-\alpha(\beta+\gamma)/2,\eta+(N+1)(N+1-\alpha+\beta+\gamma)/2,z). \tag{3.6}$$

Now relation (3.3) can be represented in the short form

$$\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}}\,\mathrm{HeunC}_N(\alpha,\beta,\gamma,\eta,z) = \mathcal{P}_N(\alpha,\beta,\gamma,\eta)\mathrm{HeunC}_N^{\maltese}(\alpha,\beta,\gamma,\eta,z), \tag{3.7}$$

where we are using the specific constant

$$\mathcal{P}_N(\alpha,\beta,\gamma,\eta) = (N+1)!\,v_{N+1}(\alpha,\beta,\gamma,-\alpha\,(N+1+(\beta+\gamma)/2)\,,\eta). \tag{3.8}$$

According to relations (2.6) we obtain $\delta_N(N+1)/\alpha(N+1)+\big(\beta(N+1)+\gamma(N+1)\big)/2 = N+1 > 0$. Thus the δ_N-condition is not fulfilled for the associated δ_N-confluent-Heun's function $\mathrm{HeunC}_N^{\maltese}(\alpha,\beta,\gamma,\eta,z)$. Hence, it does not belong to the class of the δ_N-confluent-Heun's functions.

4. A new derivation of the confluent Heun's polynomials

Now we are prepared for an alternative derivation of the confluent Heun's polynomials without using the three-term recurrence relation (1.3), i.e. directly from confluent Heun's equation. Indeed, posing the requirement

$$\mathcal{P}_N(\alpha,\beta,\gamma,\eta) = 0, \tag{4.1}$$

which is a new form of condition (1.5*b*), we obtain $\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}}\mathrm{HeunC}_N(\alpha,\beta,\gamma,\eta,z) = 0$. Thus, under condition (4.1) the δ_N-confluent-Heun's function $\mathrm{HeunC}_N(\alpha,\beta,\gamma,\eta,z)$ becomes a polynomial of degree N.[2]

[1] The explicit form relation (3.3) reads

$$\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}}\mathrm{HeunC}\left(\alpha,\beta,\gamma,-\alpha\left(\frac{\alpha+\beta}{2}+N+1\right),\eta,z\right) = (N+1)!\,v_{N+1}\left(\alpha,\beta,\gamma,-\alpha\left(\frac{\alpha+\beta}{2}+N+1\right),\eta\right)$$
$$\times\,\mathrm{HeunC}\left(\alpha,\beta+N+1,\gamma+N+1,-\alpha\frac{\beta+\gamma}{2},\eta+\frac{N+1}{2}\left(N+1-\alpha+\beta+\gamma\right),z\right)$$

for all $\alpha,\beta,\gamma,\eta\in\mathbb{C}$ (when β is not a negative integer) and for any fixed $N\in\mathbb{Z}$, $N\geqslant 0$.

[2] Note that condition (4.1) coincides with condition (1.5*b*) and its explicit form (A.2) up to a nonzero numerical factor. It can be shown that constant (3.8) is simply related to the Starobinsky constant (see [15] and [17–19]). Equation (4.1) recovers the mathematical meaning of the zero-Starobinsky-constant condition.

The explicit form (A.2) of the polynomial condition (4.1) is given in the appendix. As seen, it presents an algebraic equation of degree $(N + 1)$ for the spectral parameter μ of the confluent Heun's equation. Then from equation (A.2) we obtain $(N + 1)$-in-number roots $\mu_{k=1,\ldots,N+1}(\alpha, \beta, \gamma)$, which yield $(N + 1)$-in-number values $\eta_{k=1,\ldots,N+1}(\alpha, \beta, \gamma) = \frac{1}{2}(\alpha - \beta - \gamma + \alpha\beta - \beta\gamma) - \mu_{k=1,\ldots,N+1}(\alpha, \beta, \gamma)$ of the parameter η. Hence, condition (4.1) defines $(N + 1)$-in-number polynomial solutions

$$\mathrm{PHeunC}_{N,k}(\alpha, \beta, \gamma, z) = \mathrm{HeunC}_N(\alpha, \beta, \gamma, \eta_k, z), \qquad k = 1, \ldots, N + 1, \tag{4.2}$$

to the confluent Heun's equation, each being polynomial of degree N of the variable z.

One can find further information about the mathematical properties of the confluent Heun's polynomials $\mathrm{PHeunC}_{N,k}(\alpha, \beta, \gamma, z)$ in [5]. As a by-product we obtain that under conditions (1.5) the confluent Heun's operator $\hat{D}_{\alpha,\beta,\gamma,\delta,\eta}$ (2.1) becomes a quasi-solvable one in the sense of [20].

The associate δ_N-confluent function $\mathrm{HeunC}^{\maltese}_N(\alpha, \beta, \gamma, \eta, z)$ does not become a polynomial under condition (4.1). Indeed, the corresponding δ_N-condition is never fulfilled for it when $N \geqslant 0$—see the end of the previous section. This proofs that the function $\mathrm{HeunC}^{\maltese}_N(\alpha, \beta, \gamma, \eta, z)$ is not a polynomial when the δ_N-confluent function $\mathrm{HeunC}_N(\alpha, \beta, \gamma, \eta, z)$ is.

5. Some comments and concluding remarks

Both for the general and for the confluent hypergeometric functions a set of simple and universal representations of their repeated derivatives in terms of another hypergeometric functions are well known and widely used, see for example [21].

At present, the corresponding mathematical theory is still not developed enough to have a complete picture for this problem in the case of different Heun's functions. For general Heun's functions analogous relations seem to exist only for some *particular* cases, see for example [20, 22, 23]. In the literature known to us, one cannot find similar relations for confluent Heun's functions. In the present paper we fill this gap by partially taking into account the specific properties of the confluent Heun's functions.

The general Heun's equation, written in the universal Fuchsian form

$$\begin{gathered} H'' + \left(\frac{\gamma_G}{z} + \frac{\delta_G}{z-1} + \frac{\epsilon_G}{z-1/a}\right)H' + \frac{\alpha_G\beta_G z - q}{z(z-1)(z-1/a)}H = 0, \\ \gamma_G + \delta_G + \epsilon_G = \alpha_G + \beta_G + 1, \end{gathered} \tag{5.1}$$

was constructed by Karl Heun in [1] as a generalization of the standard hypergeometric equation, by adding one more regular singular point: $z = 1/a \in (1, \infty)$. Putting $\gamma_G = \beta + 1$, $\delta_G = \gamma + 1$, $\epsilon_G = -\alpha/a$, $\alpha_G\beta_G = -(\mu + \nu)/a$, $q = -\mu/a$, and taking the limit $a \to 0$, we obtain the confluent Heun's equation (1.2) by coalescence of the regular singular points $z = 1/a$ and $z = \infty$ in equation (5.1).

Unfortunately, such a coalescence process in the very solutions and corresponding relations is much more complicated and, as a rule, not possible. We shall illustrate this fact using the approach of [20, 23, 24]. Translating the four *different* regular singular points $z = 0, 1, 1/a, \infty$ of equation (5.1) in the compactified complex plane $\tilde{\mathbb{C}}_z$ to the places of the vortices $x = 0, 1, 1 + \tau, \tau$ (τ being pure imaginary) of the fundamental rectangle of periods in the complex plane $\tilde{\mathbb{C}}_x$ by the transformation

$$z = \frac{e_1 - e_3}{\wp(x) - e_3}, \qquad e_{1,2,3} = \wp(\omega_{1,2,3}), \tag{5.2}$$

$\wp(x)$ being the Weierstrass elliptic function with half-periods $\omega_0 = 0, \omega_1 = 1/2, \omega_2 = 1/2 + \tau/2, \omega_3 = \tau/2$, and using in addition the substitution

$$\psi(x) = z^{(l_0+1)/2}(z-1)^{(l_1+1)/2}(1-az)^{(l_2+1)/2}H(z), \qquad l_0 = \gamma_{_G} - 3/2,$$
$$l_1 = \delta_{_G} - 3/2, \qquad l_2 = \epsilon_{_G} - 3/2, \tag{5.3}$$

one can transform the general Heun's equation (5.1) into elliptic form, identical to the BC1 Inozemtsev model:

$$-\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)\psi(x) = E\psi(x). \tag{5.4}$$

Here l_3 and E are properly chosen constants. This equation is instrumental for deriving the results in [20, 23, 24], where slightly different notations are in use (see also the references therein). Obviously, equation (5.4) has no relevant limit $a \to 0$, since $\lim_{a\to 0} |l_2| = \infty$.

A more fundamental *geometrical* obstacle to relate the results for general Heun's equations and the results for the confluent one is that according to the formulae $e_3 = -\frac{2-a}{1+a}e_1, e_2 = \frac{1-2a}{1+a}e_1$ one obtains $\lim_{a\to 0} e_2 = e_1$. Hence, an elliptic representation of the confluent Heun's equation does not exist: as a result of the coalescence process, in it we have only three different singular points. An analytic map like (5.2) of the triangle of the singular points of the confluent Heun's equation onto a rectangle of the periods of elliptic functions is impossible. Hence, there exist essential differences between the properties of general and confluent Heun's functions. Note that some of the properties can be reformulated properly after the coalescence. For example, the factor $(1-az)^{(l_2+1)/2}$ has a confluent limit $\exp(\alpha z/2)$. As a result a correspondingly modified substitution (5.3) exists. It transforms the confluent Heun's equation to the Schrödinger-like (non-elliptic) form [3–7].

A direct consequence of the above consideration is the need to derive most of the basic properties of confluent Heun's functions independently of the properties of general Heun's functions. Sometimes one can use similar general methods in both cases. An example is the generalized Darboux transformation [20], used in section 3 in a specific way, and invented originally for equation (5.4).

One can hope that the new mathematical results for the confluent Heun's functions, derived in the present paper, will be instrumental in different analytical and numerical applications, and especially in the relativistic theory of gravity.

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Appendix.

The confluent Heun's equation (2.2) can be rewritten in the transparently self-adjoint form:

$$\mathrm{e}^{-\alpha z}z^{-\beta}(z-1)^{-\gamma}\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathrm{e}^{\alpha z}z^{1+\beta}(z-1)^{1+\gamma}\frac{\mathrm{d}H(z)}{\mathrm{d}z}\right) + \alpha\left(\frac{\delta}{\alpha} + \frac{\beta+\gamma}{2} + 1\right)zH(z) = \mu H(z). \tag{A.1}$$

Besides, it shows that the natural spectral parameter is μ. Correspondingly, we represent the left-hand side $\Delta_{N+1}(\mu)$ of condition (1.5*b*) in the form of the specific three-diagonal determinant:

$$\begin{vmatrix} \mu - q_1 & 1(1+\beta) & 0 & \dots & 0 & 0 & 0 \\ N\alpha & \mu - q_2 + 1\alpha & 2(2+\beta) & \dots & 0 & 0 & 0 \\ 0 & (N-1)\alpha & \mu - q_3 + 2\alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu - q_{N-1} + (N-2)\alpha & (N-1)(N-1+\beta) & 0 \\ 0 & 0 & 0 & \dots & 2\alpha & \mu - q_N + (N-1)\alpha & N(N+\beta) \\ 0 & 0 & 0 & \dots & 0 & 1\alpha & \mu - q_{N+1} + N\alpha \end{vmatrix}. \qquad \text{(A.2)}$$

It turns out to be useful in the real calculations [7]. Here $q_n = (n-1)(n+\beta+\gamma)$.

A similar representation of the second polynomial condition (1.5*b*) in determinant form was derived in [5].

Note that we can develop an alternative consideration, interchanging the places of the regular singular points $z = 0$ and $z = 1$. Then because of the obvious symmetry of equation (1.1) under the change $\{\alpha, \beta, \gamma, \delta, \mu, \nu, z\} \to \{\alpha, \gamma, \beta, \delta, \nu, \mu, z-1\}$, the spectral parameter will be $-\nu$.

References

[1] Heun K 1889 *Math. Ann.* **33** 161

[2] Bateman H and Erdélyi A 1955 *Higher Transcendental Functions* vol 3 (New York: McGraw-Hill)

[3] Decarreau A, Dumont-Lepage M Cl, Maroni P, Robert A and Roneaux A 1978 *Ann. Soc. Sci. Brux.* **92** 53

[4] Decarreau A, Maroni P and Robert A 1978 *Ann. Soc. Sci. Brux.* **92** 151

[5] Ronveaux A (ed) 1995 *Heun's Differential Equations* (New York: Oxford University Press)

[6] Slavyanov S Y and Lay W 2000 *Special Functions: A Unified Theory Based on Singularities* (Oxford: Oxford Mathematical Monographs)

[7] Fiziev P P 2009 Classes of exact solutions to Regge–Wheeler and Teukolsky equations arXiv:0902.1277

[8] Fiziev P P 2009 arXiv:0908.4234

[9] Fiziev P P 2006 *Class. Quantum Grav.* **23** 2447–68

[10] Fiziev P P 2007 *J. Phys.: Conf. Ser.* **66** 012016

[11] Fiziev P P and Staicova D R 2009 *Bulgarien Astrophys. J.* **11** 3–11

[12] Fiziev P P and Staicova D R 2009 *Bulgarien Astrophys. J.* **11** 13–21

[13] Borissov R S and Fiziev P P 2009 Exact solutions of Teukolsky master equation with continuous spectrum arXiv:0903.3617

[14] Fiziev P P 2007 Exact solutions to Regge–Wheeler and Teukolsky master equations: some applications to astrophysical problems and some open problems, Talk given at the *Seminar of AEI MPG (Golm, Germany, 26 March 2009)* http://tcpa.uni-sofia.bg/research/AEI_26_03_2009.ppt and http://numrel.aei.mpg.de/Events/Seminars/archive-huge/2008-2009/2009Mar26.fiziev.pdf

[15] Fiziev P P 2009 Teukolsky–Starobinsky identities—a novel derivation and generalizations *Phys. Rev.* D **80** 124001

[16] Batic D and Schmid H 2007 *J. Math. Phys.* **48** 042502

[17] Teukolsly S A and Press W H 1974 *Astrophys. J.* **193** 443–61

[18] Chandrasekhar S 1983 *The Mathematical Theory of Black Holes* vols 1,2 (Oxford: Oxford University Press)
[19] Chandrasekhar S 1984 *Proc. R. Soc.* A **392** 1–13
[20] Takemura T 2006 *J. Nonlinear Math. Phys.* **13** 584–611
[21] Bateman H and Erdélyi A 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
[22] Ishkhanyan A and Suominen K A 2003 *J. Phys. A: Math. Gen.* **36** L81–5
[23] Valent G 2005 Heun functions versus elliptic functions arXiv:math-ph/0512006
[24] Takemura T 2004 *Electron. J. Differ. Equ.* **15** 130